

TOROIDAL AUTOMORPHIC FORMS FOR SOME FUNCTION FIELDS

GUNTHER CORNELISSEN AND OLIVER LORSCHIED

ABSTRACT. Zagier introduced toroidal automorphic forms to study the zeros of zeta functions: an automorphic form on GL_2 is toroidal if all its right translates integrate to zero over all nonsplit tori in GL_2 , and an Eisenstein series is toroidal if its weight is a zero of the zeta function of the corresponding field. We compute the space of such forms for the global function fields of class number one and genus $g \leq 1$, and with a rational place. The space has dimension g and is spanned by the expected Eisenstein series. We deduce an “automorphic” proof for the Riemann hypothesis for the zeta function of those curves.

1. INTRODUCTION

Let X denote a smooth projective curve over a finite field \mathbf{F}_q with q elements, \mathbf{A} the adeles over its function field $F := \mathbf{F}_q(X)$, $G = GL_2$, B its standard (upper-triangular) Borel subgroup, $K = G(\mathcal{O}_{\mathbf{A}})$ the standard maximal compact subgroup of $G_{\mathbf{A}}$, with $\mathcal{O}_{\mathbf{A}}$ the maximal compact subring of \mathbf{A} , and Z the center of G . Let \mathcal{A} denote the space of unramified automorphic forms $f : G_F \backslash G_{\mathbf{A}} / K Z_{\mathbf{A}} \rightarrow \mathbf{C}$. We use the following notations for matrices:

$$\text{diag}(a, b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } \llbracket a, b \rrbracket = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

There is a bijection between quadratic separable field extensions E/F and conjugacy classes of maximal non-split tori in G_F via

$$E^{\times} = \text{Aut}_E(E) \subset \text{Aut}_F(E) \simeq G_F.$$

If T is a non-split torus in G with $T_F \cong E^{\times}$, define the space of *toroidal automorphic forms for F with respect to T (or E)* to be

$$(1) \quad \mathsf{T}_F(E) = \left\{ f \in \mathcal{A} \mid \forall g \in G_{\mathbf{A}}, \int_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} f(tg) dt = 0 \right\}.$$

The integral makes sense since $T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}$ is compact, and the space only depends on E , viz., the conjugacy class of T . The space of *toroidal automorphic forms for F* is

$$\mathsf{T}_F = \bigcap_E \mathsf{T}_F(E),$$

where the intersection is over all quadratic separable E/F . The interest in these spaces lies in the following version of a formula of Hecke ([5], Werke p. 201); see Zagier, [15] pp. 298–299 for this formulation, in which the result essentially follows from Tate’s thesis:

Proposition 1.1. *Let ζ_E denote the zeta function of the field E . Let $\varphi : \mathbf{A}^2 \rightarrow \mathbf{C}$ be a Schwartz-Bruhat function. Set*

$$f(g, s) = |\det g|_F^s \int_{\mathbf{A}^{\times}} \varphi((0, a)g) |a|^{2s} d^{\times} a.$$

An Eisenstein series $E(s)$

$$E(s)(g) := \sum_{\gamma \in B_F \backslash G_F} f(\gamma g, s) \quad (\operatorname{Re}(s) > 1)$$

satisfies

$$\int_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} E(s)(tg) dt = c(\varphi, g, s) |\det g|^s \zeta_E(s)$$

for some holomorphic function $c(\varphi, g, s)$. For every g and s , there exists a function φ such that $c(\varphi, g, s) \neq 0$. In particular, $E(s) \in \mathbb{T}_F(E) \iff \zeta_E(s) = 0$. \square

Remark 1.2. Toroidal integrals of parabolic forms are ubiquitous in the work of Waldspurger ([13], for recent applications, see Clozel and Ullmo [1] and Lysenko [10]). Wielonsky and Lachaud studied analogues for GL_n , $n \geq 2$, and tied up the spaces with Connes' view on zeta functions ([14], [7], [6], [2]).

Let $\mathcal{H} = C_0^\infty(K \backslash G_{\mathbf{A}}/K)$ denote the bi- K -invariant Hecke algebra, acting by convolution on \mathcal{A} . There is a correspondence between K -invariant $G_{\mathbf{A}}$ -modules and Hecke modules; in particular, we have

Lemma 1.3. The spaces $\mathbb{T}_F(E)$ (for each E with corresponding torus T) and \mathbb{T}_F are invariant under the Hecke algebra \mathcal{H} , and

$$(2) \quad \mathbb{T}_F(E) \subseteq \left\{ f \in \mathcal{A} \mid \forall \Phi \in \mathcal{H}, \int_{T_F Z_{\mathbf{A}} \backslash T_{\mathbf{A}}} \Phi(f)(t) dt = 0 \right\}. \quad \square$$

Now assume F has class number one and there exists a place ∞ of degree one for F ; let t denote a local uniformizer at ∞ . Strong approximation implies that we have a bijection

$$G_F \backslash G_{\mathbf{A}}/KZ_\infty \xrightarrow{\sim} \Gamma \backslash G_\infty/K_\infty Z_\infty,$$

where $\Gamma = G(A)$ with A the ring of functions in F holomorphic outside ∞ , and a subscript ∞ refers to the ∞ -component. We define a graph \mathcal{T} with vertices $V\mathcal{T} = G_\infty/K_\infty Z_\infty$. If \sim denotes equivalence of matrices modulo $K_\infty Z_\infty$, then we call vertices in $V\mathcal{T}$ given by classes represented by matrices g_1 and g_2 adjacent, if $g_1^{-1}g_2 \sim \begin{bmatrix} t & b \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} t^{-1} & 0 \\ 0 & 1 \end{bmatrix}$ for some $b \in \mathcal{O}_\infty/t$. Then \mathcal{T} is a tree that only depends on q (the so-called Bruhat-Tits tree of $\mathrm{PGL}(2, F_\infty)$, cf. [11], Ch. II).

The Hecke operator $\Phi_\infty \in \mathcal{H}$ given by the characteristic function of $K[[t, 0]]K$ maps a vertex of \mathcal{T} to its neighbouring vertices. The action of Φ_∞ on the quotient graph $\Gamma \backslash \mathcal{T}$ can be computed from the orders of the Γ -stabilizers of vertices and edges in \mathcal{T} . When drawing a picture of $\Gamma \backslash \mathcal{T}$, we agree to label a vertex along the edge towards an adjacent vertex by the corresponding weight of a Hecke operator, as in the next example.

Example 1.4. In Figure 1, one sees the graph $\Gamma \backslash \mathcal{T}$ for the function field of $X = \mathbf{P}^1$, with the well-known vertices representing $\{c_i = [\pi^{-i}, 0]\}_{i \geq 0}$ and the weights of Φ_∞ , meaning

$$(3) \quad \text{for } n \geq 1, \Phi_\infty(f)(c_n) = qf(c_{n-1}) + f(c_{n+1}) \text{ and } \Phi_\infty(f)(c_0) = (q+1)f(c_1).$$

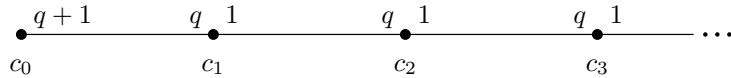


FIGURE 1. The graph $\Gamma \backslash \mathcal{T}$ for $X = \mathbf{P}^1$

Further useful facts: One easily calculates that $X_q(\mathbf{F}_{q^2})$ is cyclic of order $2q + 1$; let Q denote any generator. We will use later on that the vertices t_i correspond to classes of rank-two vector bundles on $X_q(\mathbf{F}_q)$ that are pushed down from line bundles on $X_q(\mathbf{F}_{q^2})$ given by multiples $Q, 2Q, \dots, qQ$ of Q , cf. Serre, loc. cit. For a representation in terms of matrices, one may refer to [12]: if $iQ = (\ell, *) \in X_q(\mathbf{F}_{q^2})$, then $t_i = \llbracket t^2, t^{-1} + \ell t \rrbracket$.

We denote a function f on $\Gamma \backslash \mathcal{T}$ by a vector

$$f = [f(t_1), \dots, f(t_q) \mid f(z_0), f(z_1) \mid f(c_0), f(c_1), f(c_2), \dots].$$

Proposition 3.1. *A function $f \in \mathbb{T}_{F_q}(F_q^{(2)})$ ($q = 2, 3, 4$) belongs to the Φ_∞ -stable linear space \mathcal{S} of functions*

$$(4) \quad \mathcal{S} := \{ [T_1, \dots, T_q \mid Z_0, Z_1 \mid C_0, C_1, C_2, \dots] \}$$

with $C_0 = -2(T_1 + \dots + T_q)$ and for $k \geq 0$,

$$(5) \quad C_k = \begin{cases} \lambda_k Z_0 + \mu_k (T_1 + \dots + T_q) & \text{if } k \text{ even} \\ \nu_k Z_1 & \text{if } k \text{ odd} \end{cases}$$

for some constants λ_k, μ_k, ν_k . In particular,

$$\dim \mathbb{T}_{F_q}(F_q^{(2)}) \leq \dim \mathcal{S} = q + 2,$$

and $\dim \mathbb{T}_{F_q}$ is finite.

Proof. We choose arbitrary values T_j at t_j ($j = 1, \dots, q$) and Z_j at z_j ($j = 1, 2$), and set $\tau = T_1 + \dots + T_q$. We have

$$\int_{T_F Z_A \backslash T_A} f(t) dt = C_0 + 2\tau.$$

Indeed, by the same reasoning as in the proof of Theorem 2.1, the integration area maps to the image of

$$\text{Pic}(X_q(\mathbf{F}_{q^2}))/\text{Pic}(X_q(\mathbf{F}_q)) = X_q(\mathbf{F}_{q^2})/X_q(\mathbf{F}_q) = X_q(\mathbf{F}_{q^2})$$

(the final equality since X_q is assumed to have class number one) in $\Gamma \backslash \mathcal{T}$, and these are exactly the vertices c_0 and t_j (the latter with multiplicity two, since $\pm Q \in E(\mathbf{F}_{q^2})$ map to the same vertex). The integral is zero exactly if $C_0 = -2\tau$. Applying the Hecke operator Φ_∞ to this equation (cf. (2)) gives $C_1 = -2Z_1$, then applying Φ_∞ again gives $C_2 = -(q+1)Z_0$. The rest follows by induction. If we apply Φ_∞ to the equations (5) for $k \geq 2$, we find by induction for k even that

$$C_{k+1} = \lambda_k C_1 + (\lambda_k q + \mu_k q(q+1) - q\nu_{k-1})Z_1$$

and for k odd that

$$C_{k+1} = (\nu_k - q\lambda_{k-1})Z_0 + (\nu_k - q\mu_{k-1})\tau.$$

□

Lemma 3.2. *The space \mathcal{S} from (4) has a basis of $q + 2$ Φ_∞ -eigenforms, of which exactly $q - 1$ are cusp forms with eigenvalue zero and support in the set of vertices $\{t_j\}$, and three are non-cuspidal forms with respective eigenvalues $0, q, -q$.*

Proof. With $\tau = T_1 + \dots + T_q$, the function

$$f = [T_1, \dots, T_q \mid Z_0, Z_1 \mid -2\tau, C_1, C_2, \dots]$$

is a Φ_∞ -eigenform with eigenvalue λ if and only if

$$\lambda T_j = (q+1)Z_1; \lambda Z_1 = \tau + Z_0; \lambda Z_0 = qZ_1 + C_1; \lambda(-2\tau) = (q+1)C_1; \text{ etc.}$$

We consider two cases:

(a) if $\lambda = 0$, we find q forms

$$f_k = [0, \dots, 0, 1, 0, \dots, 0 \mid 0, -1 \mid -q, \dots]$$

with $T_j = 1 \iff j = k$.

(b) if $\lambda \neq 0$, we find $\lambda = \pm q$ with eigenforms

$$f_{\pm} = [q + 1, \dots, q + 1 \mid -q, \pm q \mid -2q(q + 1), \mp 2q^2, \dots].$$

Since we found $q + 2$ eigenforms, they span \mathcal{S} . From the fact that a cusp form satisfies $f(c_i) = 0$ for all i sufficiently large (cf. Harder [4], Thm. 1.2.1), one easily deduces that a basis of cusp forms in \mathcal{S} consists of $f_k - f_1$ for $k = 2, \dots, q$. \square

Corollary 3.3. *The Riemann hypothesis is true for ζ_{F_q} ($q = 2, 3, 4$).*

Proof. From Lemma 3.2, we deduce that the only possible Φ_{∞} -eigenvalue of a toroidal Eisenstein series is $\pm q$ or 0, but on the other hand, from Lemma 1.1, we know this eigenvalue is $q^s + q^{1-s}$ where $\zeta_{F_q}(s) = 0$. We deduce easily that s has real part $1/2$. \square

Remark 3.4. One may verify that this proves the Riemann Hypothesis for the fields F_q without actually computing ζ_{F_q} : it only uses the expression for the zeta function by a Tate integral. Using a sledgehammer to crack a nut, one may equally deduce from Theorem 2.1 that $\zeta_{\mathbf{P}_1}$ doesn't have any zeros. At least the above corollary shows how enough knowledge about the space of toroidal automorphic forms does allow one to deduce a Riemann Hypothesis, in line with a hope expressed by Zagier [15].

Theorem 3.5. *For $q = 2, 3, 4$, \mathbb{T}_{F_q} is one-dimensional, spanned by the Eisenstein series of weight s equal to a zero of the zeta function ζ_{F_q} of F_q .*

Remark 3.6. Note that the functional equation for $E(s)$ implies that $E(s)$ and $E(1 - s)$ are linearly dependent, so it doesn't matter which zero of ζ_{F_q} is taken.

Proof. By Lemma 3.2, \mathbb{T}_{F_q} is a Φ_{∞} -stable subspace of the finite dimensional space \mathcal{S} , and Φ_{∞} is diagonalizable on \mathcal{S} . By linear algebra, the restriction of Φ_{∞} is also diagonalizable on \mathbb{T}_{F_q} with a subset of the given eigenvalues, hence \mathbb{T}_{F_q} is a subspace of the space of automorphic forms for the corresponding eigenvalues of Φ_{∞} . By [8], Theorem 7.1, it can therefore be split into a direct sum of a space of Eisenstein series \mathcal{E} , a space of residues of Eisenstein series \mathcal{R} , and a space of cusp forms \mathcal{C} (note that in the slightly different notations of [8], “residues of Eisenstein series” are called “Eisenstein series”, too). We treat these spaces separately.

\mathcal{E} : By Proposition 1.1, $\mathbb{T}_{F_q}(F_q^{(2)})$ contains exactly two Eisenstein series, one corresponding to a zero s_0 of ζ_{F_q} , and one corresponding to a zero s_1 of

$$L_q(s) := \zeta_{F_q^{(2)}}(s) / \zeta_{F_q}(s).$$

Now consider the torus \tilde{T} corresponding to the quadratic extension $E_q = F_q(z)/F_q$ of genus two defined by $x = z(z + 1)$. Set

$$\tilde{L}_q(s) := \zeta_{E_q}(s) / \zeta_{F_q}(s)$$

and $T = q^{-s}$. One computes immediately that $L_q = qT^2 + qT + 1$ but

$$\tilde{L}_2 = 2T^2 + 1, \tilde{L}_3 = 3T^2 + T + 1 \text{ and } \tilde{L}_4 = 4T^2 + 1.$$

Since L_q and \tilde{L}_q have no common zero, the \tilde{T} -integral of the Eisenstein series of weight s_1 is non-zero, and hence it doesn't belong to \mathbb{T}_{F_q} . Hence \mathcal{E} is as expected.

\mathcal{R} : Elements in \mathcal{R} have Φ_{∞} -eigenvalues $\neq 0, \pm q$, so cannot even occur in \mathcal{S} : since the class number of F_q is one, \mathcal{R} is spanned by the two forms

$$r_{\pm} := [1, \dots, 1 \mid \pm 1, 1 \mid 1, \pm 1, 1, \pm 1, \dots]$$

with $r(c_i) = (\pm 1)^i$, and this is a Φ_{∞} -eigenform with eigenvalue $\pm(q + 1)$. (In general, the space is spanned by elements of the form $\chi \circ \det$ with χ a class group character, cf. [3], p. 174.)

\mathcal{C} : By multiplicity one, \mathcal{C} has a basis of simultaneous \mathcal{H} -eigenforms. From Lemma 3.2, we know that potential cusp forms in \mathbb{T}_{F_q} have support in the set of vertices $\{t_i\}$. To prove that $\mathcal{C} = \{0\}$, the following therefore suffices:

Proposition 3.7. *The only cusp form which is a simultaneous eigenform for the Hecke algebra \mathcal{H} and has support in $\{t_i\}$ is $f = 0$.*

Proof. Let f denote such a form. Fix a vertex $\mathfrak{t} \in \{t_i\}$. It corresponds to a point $P = (\ell, *)$ on $X_q(\mathbf{F}_{q^2})$, which is a place of degree two of $\mathbf{F}_q(X_q)$. Let Φ_P denote the corresponding Hecke operator. We claim that

Lemma 3.8. $\Phi_P(c_0) = (q+1)c_2 + q(q-1)\mathfrak{t}$.

Given this claim, we finish the proof as follows: we assume that f is a Φ_P -eigenform with eigenvalue λ_P . Then

$$0 = \lambda_P f(c_0) = \Phi_P f(c_0) = q(q-1)f(\mathfrak{t}) + (q+1)f(c_2) = q(q-1)f(\mathfrak{t})$$

since $f(c_2) = 0$, hence $f(\mathfrak{t}) = 0$ for all \mathfrak{t} .

Proof of Lemma 3.8 As in [3], 3.7, the Hecke operator Φ_P maps the identity matrix (= the vertex c_0) to the set of vertices corresponding to the matrices $m_\infty := \text{diag}(\pi, 1)$ and $m_b := \begin{pmatrix} 1 & b \\ 0 & \pi \end{pmatrix}$, where $\pi = x - \ell$ is a local uniformizer at P and b runs through the residue field at P , which is

$$\mathbf{F}_q[X_q]/(x - \ell) = \mathbf{F}_q[y]/F(\ell, y) \cong \mathbf{F}_{q^2}$$

if $F(x, y) = 0$ is the defining equation for X_q . Hence we can represent every such b as $b = b_0 + b_1 y$. We now reduce these matrices to a standard form in $\Gamma \backslash \mathcal{S}$ from [12], §2. By right multiplication with $\llbracket 1, -b_0 \rrbracket$, we are reduced to considering only $b = b_1 y$.

If $b_1 = 0$, then the matrix is $m_b = \text{diag}(1, \pi) \sim \text{diag}(\pi^{-1}, 1)$. Recall that $t = x/y$ is a uniformizer at ∞ , so $x - \ell = t^{-2} \cdot A$ for some $A \in \mathbf{F}_q[[t]]^*$. Hence right multiplication by $\text{diag}(A^{-1}, 1)$ gives that this matrix reduces to c_2 . The same is true for m_∞ .

On the other hand, if $b_1 \neq 0$, multiplication on the left by $\text{diag}(1, b_1)$ and on the right by $\text{diag}(1, b_1^{-1})$ reduces us to considering m_y . By multiplication on the right with

$$\text{diag}((x - \ell)^{-1} \cdot A, (x - \ell)^{-1}),$$

we get $m_y \sim \llbracket t^2, y/(x - \ell) \rrbracket$. Now note that

$$\frac{y}{x - \ell} = \frac{y}{x} \cdot \left(1 + \frac{\ell}{x} + \left(\frac{\ell}{x} \right)^2 + \dots \right) = t^{-1} + \ell t + \beta(t) t^2$$

for some $\beta \in \mathbf{F}_q[[t]]$. Hence right multiplication with $\llbracket 1, -\beta \rrbracket$ gives $m_y \sim \llbracket t^2, t^{-1} + \ell t \rrbracket$, and this is exactly the vertex \mathfrak{t} . \square

Remark 3.9. Using different methods, more akin the geometrical Langlands programme, the second author ([9]) has generalized the above results as follows. For a general function field F of genus g and class number h , one may show that \mathbb{T}_F is finite dimensional. Its Eisenstein part is of dimension at least $h(g-1) + 1$. Residues of Eisenstein series are never toroidal. For general elliptic function fields, there are no toroidal cusp forms. For a general function field, the analogue of a result of Waldspurger ([13], Prop. 7) implies that the cusp forms in \mathbb{T}_F are exactly those having vanishing central L -value.

REFERENCES

- [1] Laurent Clozel and Emmanuel Ullmo. Équidistribution de mesures algébriques. *Compos. Math.*, 141(5):1255–1309, 2005.
- [2] Alain Connes. Trace formula in noncommutative geometry and the zeros of the Riemann zeta function. *Selecta Math. (N.S.)*, 5(1):29–106, 1999.
- [3] Stephen S. Gelbart. *Automorphic forms on adele groups*. Annals of Math. Studies, vol. 83, P.U.P., Princeton, 1975.
- [4] Günter Harder. Chevalley groups over function fields and automorphic forms. *Ann. of Math.* 100(2):249–306, 1974.
- [5] Erich Hecke. Über die Kroneckersche Grenzformel für reelle quadratische Körper und die Klassenzahl relativ-abelscher Körper. *Verhandlungen der Naturforschenden Gesellschaft in Basel* 28:363–372, 1917 = Werke, pp. 198–207 (no. 10).
- [6] Gilles Lachaud. Zéros des fonctions L et formes toriques. *C. R. Math. Acad. Sci. Paris*, 335(3):219–222, 2002.
- [7] Gilles Lachaud. Spectral analysis and the Riemann hypothesis. *J. Comput. Appl. Math.*, 160(1-2):175–190, 2003.
- [8] Wen-Ch’ing Winnie Li. Eisenstein series and decomposition theory over function fields. *Math. Ann.* 240:115–139, 1979.
- [9] Oliver Lorscheid. Toroidal automorphic forms for function fields. Ph.D. Thesis, Utrecht University, 2008.
- [10] Sergei Lysenko. Geometric Waldspurger periods. preprint math.AG/0510110 (2005).
- [11] Jean-Pierre Serre. *Trees*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.
- [12] Shuzo Takahashi. The fundamental domain of the tree of $GL(2)$ over the function field of an elliptic curves. *Duke Math. J.* 72(1):85–97, 1993.
- [13] Jean-Loup Waldspurger. Sur les valeurs de certaines fonctions L automorphe en leur centre de symétrie. *Compos. Math.* 54:173–242, 1985.
- [14] Franck Wielonsky. Séries d’Eisenstein, intégrales toroïdales et une formule de Hecke. *Enseign. Math. (2)*, 31(1-2):93–135, 1985.
- [15] Don Zagier. Eisenstein series and the Riemann zeta function. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pp. 275–301.

MATHEMATISCH INSTITUUT, UNIVERSITEIT UTRECHT, POSTBUS 80.010, 3508 TA UTRECHT, NEDERLAND
E-mail address: {cornelis,lorschei}@math.uu.nl